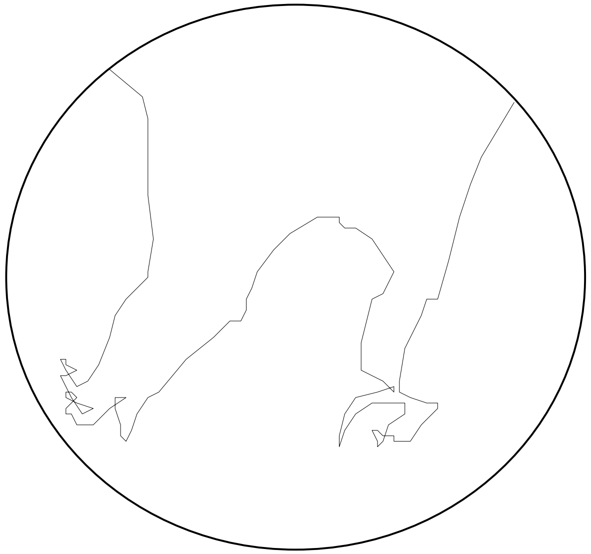
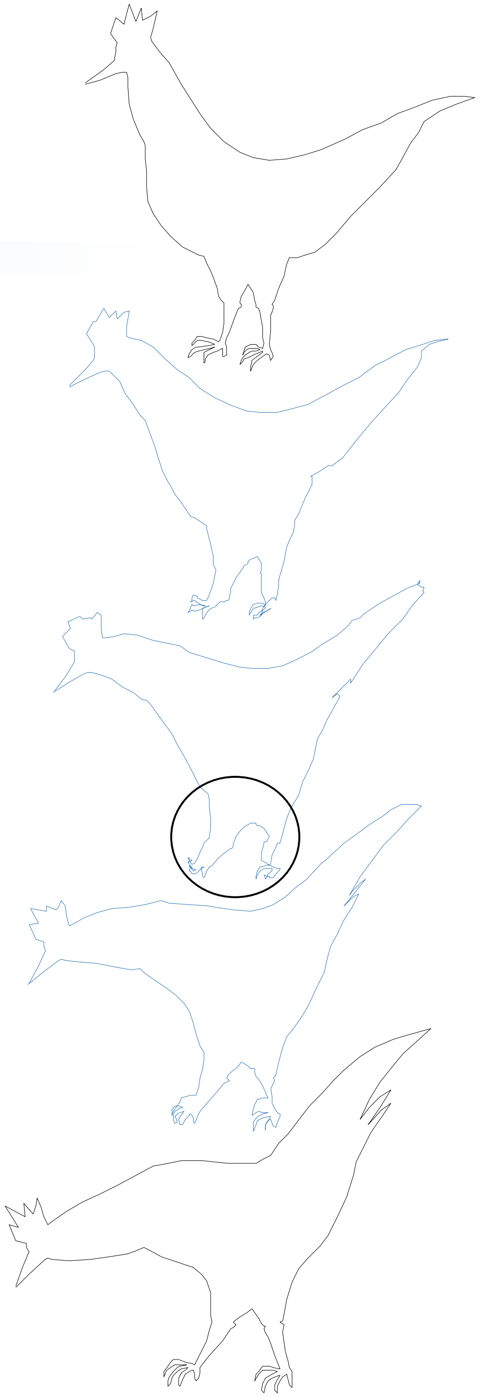
Polar Decomposition

CS418 Interactive Computer Graphics
John C. Hart

Keyframe Animation

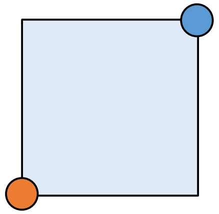
- Set target positions for vertices at “key” frames in animations
- Linearly interpolate vertex positions between targets at intervening frames
- Lots can go wrong (like the feet)





A motivating example from: Sederberg & Greenwood, A Physically-Based Approach to 2-D Shape Blending, Proc. SIGGRAPH 92

Keyframe Animation



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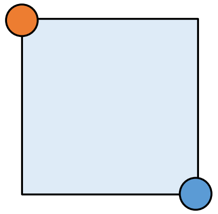

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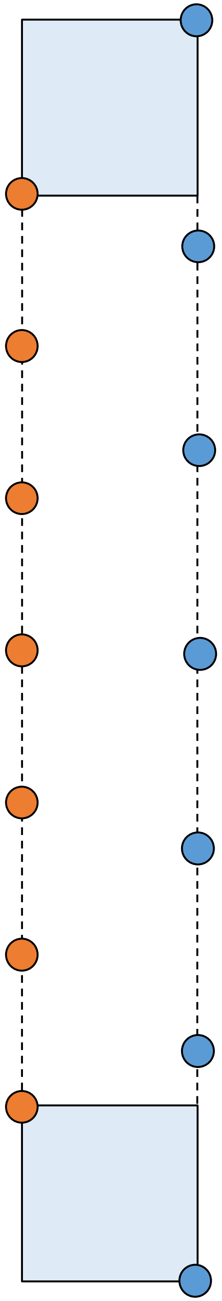

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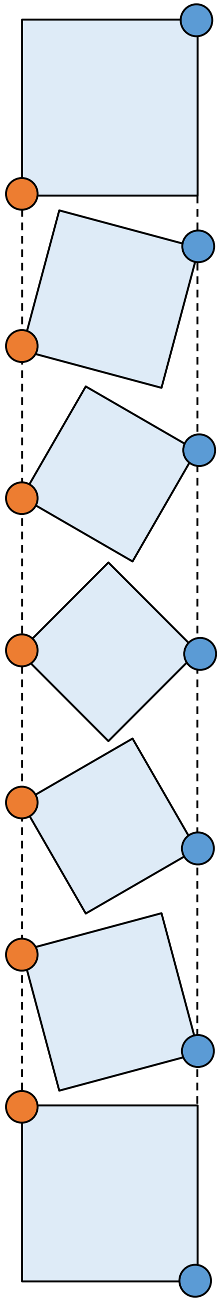
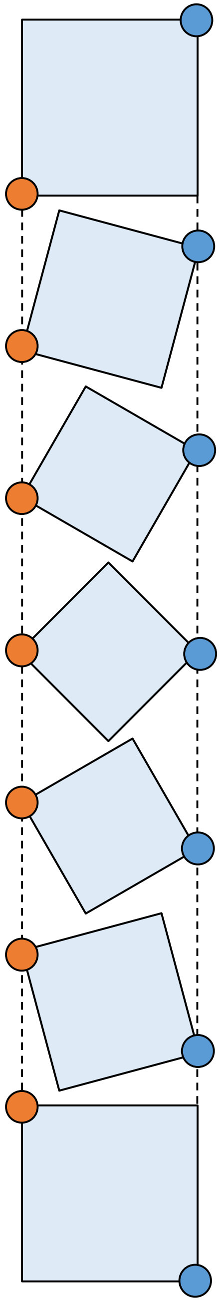


Keyframe Animation

- Set target positions for vertices at “key” frames in animations
- Linearly interpolate vertex positions between targets at intervening frames
- Shape changes size in intermediate frames of animation



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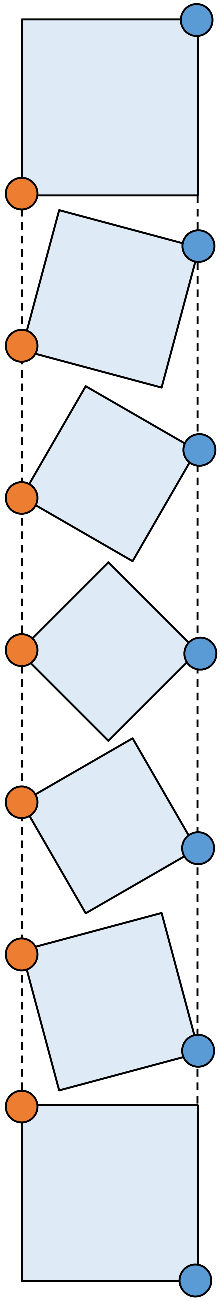

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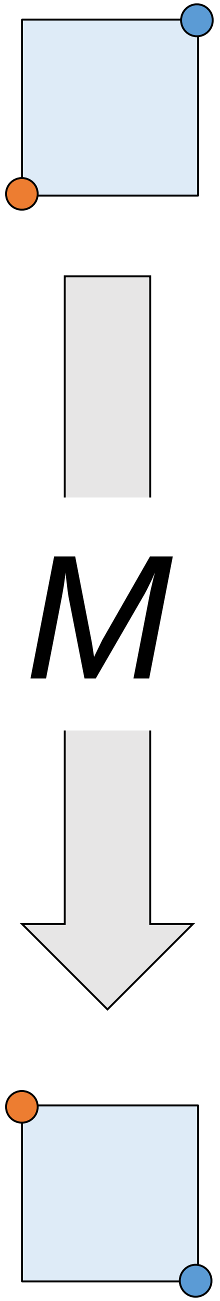

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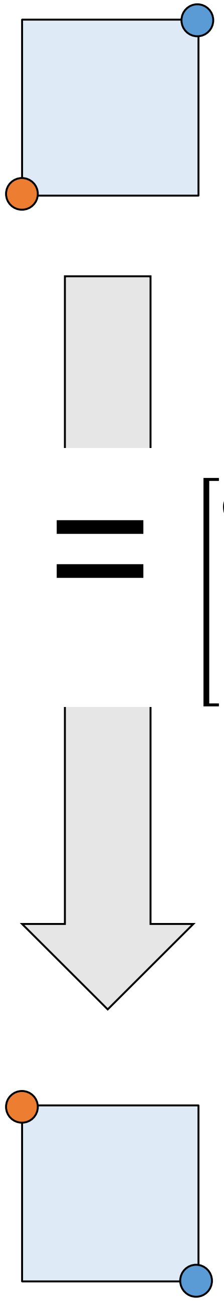


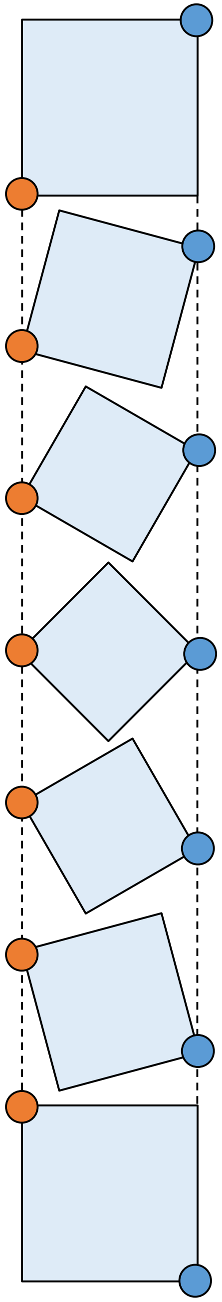
$$M = \begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & -d \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Polar Decomposition

- Let M be the upper-left 3x3 submatrix of a 4x4 homogeneous transformation matrix

$$M = \begin{bmatrix} 1/2 & 1/2 & \\ -1/2 & 1/2 & \\ & & 1 \end{bmatrix}$$

- Decompose: $M = QS$
  - Q: non-linearly varying part (rotation)
  - S: linearly varying part (scale, shear)



$$\begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

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Polar Decomposition

- Initialize $Q = M$

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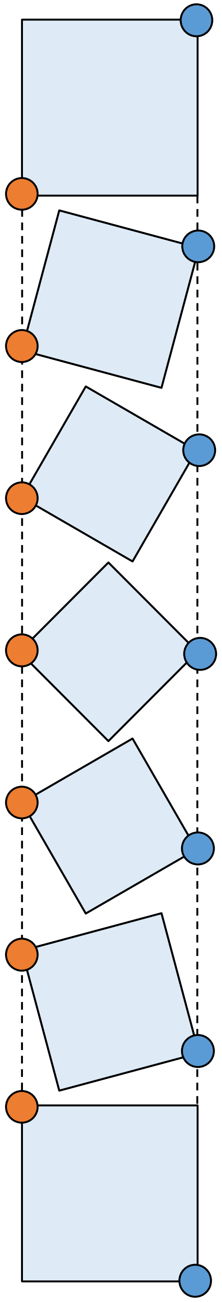

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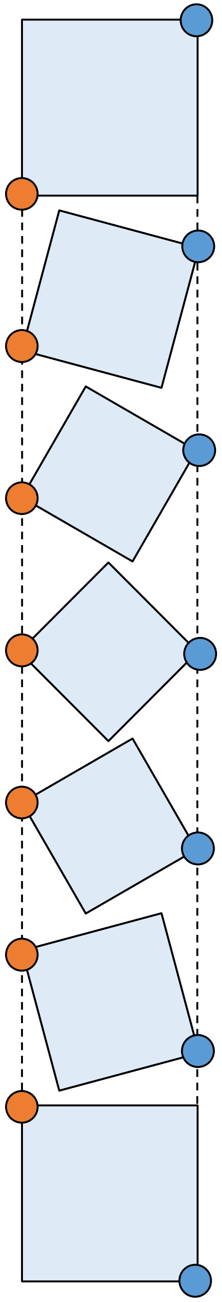
Polar Decomposition

- Initialize $Q = M$

$$Q = \begin{bmatrix} 1/2 & 1/2 & \\ -1/2 & 1/2 & \\ & & 1 \end{bmatrix}$$

- Replace $Q = ½\,(Q + Q^{-T})$



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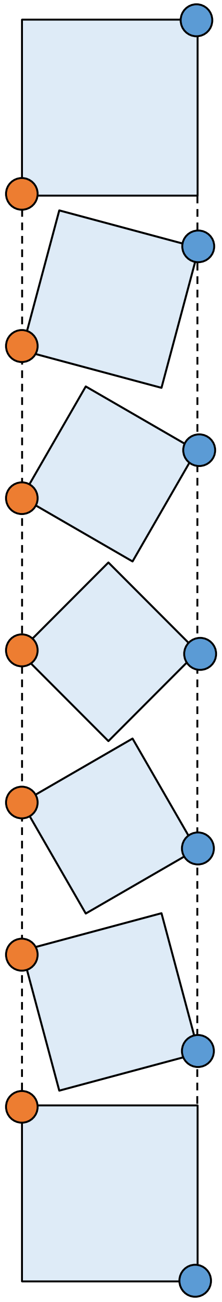
$$Q = 1/2 \left(\begin{bmatrix} 1/2 & 1/2 & \\ -1/2 & 1/2 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & \\ -1 & 1 & \\ & & 1 \end{bmatrix} \right)$$



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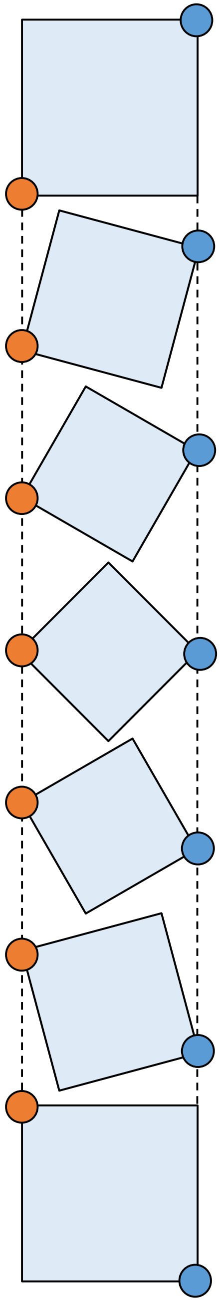
$$= \begin{bmatrix} 3/4 & 3/4 & \\ -3/4 & 3/4 & \\ & & 1 \end{bmatrix}$$



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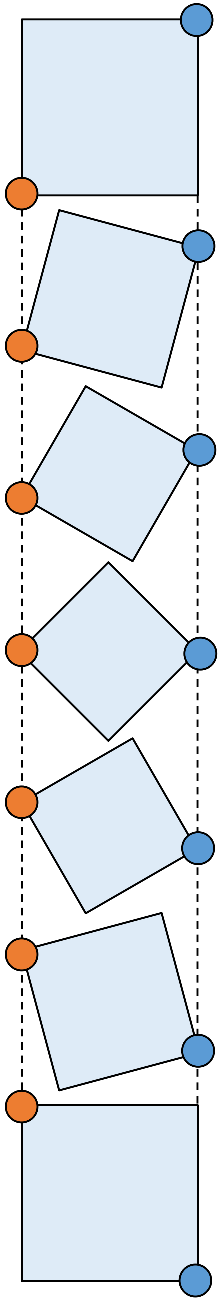
$$= \begin{bmatrix} 3/4 & 3/4 & \\ -3/4 & 3/4 & \\ & & 1 \end{bmatrix}$$

- Repeate until it convergence to a 3x3 rotation matrix ($Q^T = Q^{-1}$)

$$\begin{bmatrix} .707 & .707 & \\ -.707 & .707 & \\ & & 1 \end{bmatrix}$$



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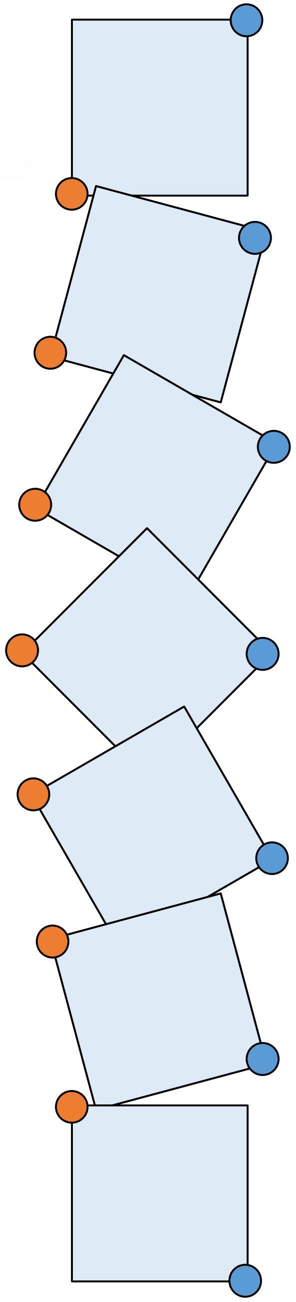

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Polar Decomposition

- "Divide" M by Q to get S

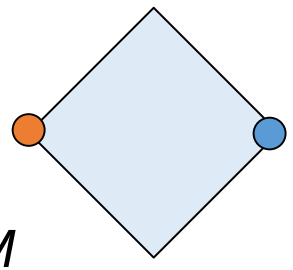
$M = QS$

$S = Q^{-1} M = Q^{T} M$

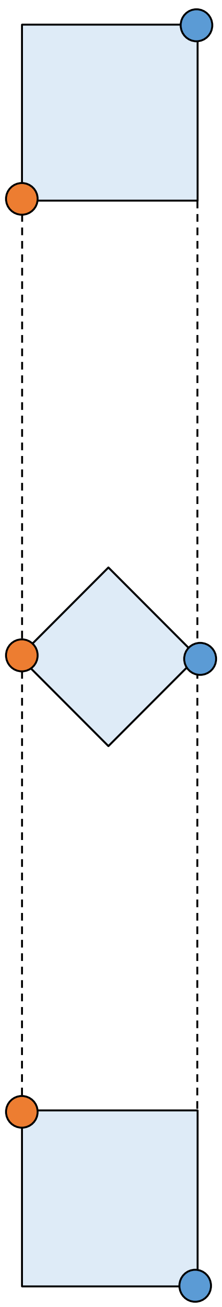
$$M = \begin{bmatrix} 1/2 & 1/2 & \\ -1/2 & 1/2 & \\ & & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} .707 & .707 & \\ -.707 & .707 & \\ & & 1 \end{bmatrix}$$

$S = Q^{T}M$



$$= \begin{bmatrix} .707 & & \\ & .707 & \\ & & 1 \end{bmatrix}$$



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